

# Exploring extra dimensions with scalar waves

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This paper provides a pedagogical introduction to the physics of extra dimensions focussing on the ADD, Randall-Sundrum and DGP models. In each of these models, the familiar particles and fields of the standard model are assumed to be confined to a four dimensional space-time called the brane; the brane is a slice through a higher dimensional space-time called the bulk. The geometry of the ADD, Randall-Sundrum and DGP space-times is described and the relation between Randall-Sundrum and Anti-de-Sitter space-time is explained. The necessary differential geometry background is introduced in an appendix that presumes no greater mathematical preparation than multivariable calculus. The ordinary wave equation and the Klein-Gordon equation are briefly reviewed followed by an analysis of the propagation of scalar waves in the bulk in all three extra-dimensional models. We also calculate the scalar field produced by a static point source located on the brane for all three models. For the ADD and Randall-Sundrum models at large distances the field looks like that of a point source in four space-time dimensions but at short distances it crosses over to a form appropriate to the higher dimensional space-time. For the DGP model the field has the higher dimensional form at long distances rather than short. The scalar field results provide qualitative insights into the corresponding behavior of gravitational fields. In particular the explanation within the ADD and Randall-Sundrum model of the weakness of gravity compared to other forces is discussed as are the implications of the two models for colliders and other experiments. The cosmological implications of the DGP model for the expansion of the Universe are briefly mentioned. Theories of extra dimensions sit at the intersection of some of the most exciting developments in theoretical and experimental physics today that we discuss in the conclusion. The material presented in this article should be accessible after one semester courses in electromagnetism and quantum mechanics and may be incorporated in such courses as well as introductory courses in particle physics and general relativity. To make the article more useful for instructors as well as for self-study a set of problems is provided in an appendix.

## I. INTRODUCTION

On the basis of everyday experience it seems evident that the space we inhabit is three dimensional. Indeed on all scales that have been probed thus far it appears that space-time is four dimensional (three dimensions of space and one time dimension). Nonetheless it remains a possibility that additional hidden dimensions do exist. A fifth dimension was invoked by Kaluza and Klein in the 1920s in order to develop a unified theory of electromagnetism and gravity. Klein [1] introduced the hypothesis that the extra dimension was rolled up, with an estimated circumference of order  $10^{-32}$  m, too small to be directly observed by experiments either then or now. The idea of extra dimensions was revived in the 1980s by the discovery that superstring theory is only consistent in ten space-time dimensions [2]. At first string theorists presumed that the extra dimensions were rolled up with a circumference on the Planck scale ( $\ell_P = 1.6 \times 10^{-35}$  m), and hence unobservable, but in 1998 Arkani-Hamed, Dimopoulos and Dvali (ADD) showed extra dimensions could be much larger and yet remain hidden [3]. Moreover sufficiently large extra dimensions could explain the hierarchy problem: the weakness of gravity compared to other forces or, equivalently, the lightness of elementary particles compared to the Planck mass scale. Randall and Sundrum then showed that a warped extra dimension provided a different and more appealing explanation of the hierarchy problem [4]; remarkably since their extra

dimension was warped it could be infinite (more precisely non-compact) and yet hidden [5]. Somewhat later Dvali, Gabadadze and Porrati (DGP) invented another model with an infinite extra dimension that sought to explain the observed accelerated expansion of the Universe rather than the hierarchy problem [6]. In all of these models the familiar four dimensional space-time inhabited by particles and fields of the standard model is a brane: a slice through a higher dimensional universe called the bulk. Excitement about extra dimensions remains high. The Large Hadron Collider has begun a second run at two times the energy of the first run that yielded the Higgs boson. At these energies it will continue to search for extra dimensions [7]. Cosmology data sets probe physics on even higher energy scales albeit indirectly [8]. Cosmology also probes gravity on the longest length scales [9]. Table top and solar system tests of gravity, astrophysical observations and precision atomic experiments also powerfully constrain extra dimensions [7, 9]. Brane worlds with extra dimensions have also engendered tremendous public interest. Randall's *Warped Passages* [10] was a bestseller and a version of the Randall-Sundrum model was the scientific basis of the five dimensional world depicted in Christopher Nolan's 2014 movie *Interstellar* [11].

The purpose of this article is to provide a pedagogical introduction to this subject at a level accessible to undergraduate physics majors. Our intention is to fill a gap between popular accounts, such as refs [10–12], and review articles, such as [9, 13], that require advanced graduate

coursework in general relativity, field theory and standard model physics. The target audience for our paper includes physicists in other fields as well as undergraduate and graduate students. No specialized background is required beyond one semester courses in electromagnetism and quantum mechanics. Indeed the material in this paper could be incorporated into such courses as well as introductory courses on particle physics or general relativity. In order to make our article more useful for instructors, as well as for self study, we provide a number of problems in appendix C.

## II. MINKOWSKI SPACE-TIME

Minkowski space-time is the staging ground for the standard model of particle physics. The geometry of Minkowski space-time is fully described by the observation that in Cartesian co-ordinates the space-time interval between two neighboring events, with co-ordinates  $(x, y, z, t)$  and  $(x + dx, y + dy, z + dz, t + dt)$  respectively, is given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad (1)$$

(see for example ref [14], vol I, chapter 17). The co-ordinates have the range  $-\infty < x, y, z, t < \infty$  and we are working in a system of units wherein the speed of light  $c = 1$ . In order to understand the physics of Minkowski space-time for simplicity we will study the propagation of scalar waves therein. This will give us insights into the propagation of electromagnetic waves (with which the reader is likely familiar) as well as gravitational waves. We will then repeat this exercise in the less familiar space-times of the ADD, Randall-Sundrum and DGP models.

A scalar wave propagating in Minkowski space-time obeys the familiar wave equation

$$\square^2 \phi = \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \phi = s(\mathbf{r}, t). \quad (2)$$

Here  $\phi$  is the scalar field and  $s$  is its source. In the absence of sources the wave equation has the plane wave solutions  $\phi = f(\mathbf{r}; \mathbf{k}) \exp(i\omega t)$  where the mode function

$$f(\mathbf{r}; \mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (3)$$

and the frequency  $\omega$  and wave-vector  $\mathbf{k}$  obey the dispersion relation  $\omega = k$ .

Turning to the sourced wave equation for simplicity we limit attention to the case of static sources  $s(\mathbf{r})$  that do not depend on time. It follows that the field  $\phi$  will also be independent of time. We consider first a point source  $s(\mathbf{r}) = \lambda \delta(\mathbf{r})$ ; more general static solutions can be developed by superposition. In effect we are just solving Poisson's equation with a delta function source and so the answer can immediately be written down from the

known electrostatic potential of a point charge. However for later use it is helpful to describe another approach. Taking the Fourier transform with respect to the position dependence in eq (2) yields  $k^2 \tilde{\phi}(\mathbf{k}) = \lambda$ ; here  $\phi$  is the Fourier transform of  $\phi$ . Inverting the Fourier transform yields

$$\phi(\mathbf{r}) = \lambda \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{k^2} \exp(i\mathbf{k} \cdot \mathbf{r}) = \frac{\lambda}{4\pi r}. \quad (4)$$

The Fourier integral is most easily evaluated by working in spherical polar co-ordinates and integrating over the angular variables first. The radial integral can be performed making use of  $\int_0^\infty dx (\sin x)/x = \pi/2$ .

The parallels to electromagnetism should be clear. In the absence of sources Maxwell's equations have plane wave solutions analogous to eq (3); these solutions famously led Maxwell to identify light as an electromagnetic phenomenon. And an electrostatic point charge produces a  $1/r$  potential just as a scalar point source does. The differences from electromagnetism should also be clear. Electromagnetic waves have polarization and electromagnetic fields are sourced by currents as well as charges. In the potential formulation the electromagnetic field is a 4-vector not a scalar. Courses in electromagnetism cover the scalar wave equation as a prelude to—not a substitute for—Maxwell's equations (see for example ref [14], vol II, chapter 20). Nonetheless the scalar wave equation is a useful caricature of Maxwell's equations and its solutions do provide insights into the full electromagnetic theory.

The parallels to gravity may be less familiar but are also present. In the absence of sources Einstein's equations have plane wave solutions analogous to eq (3) discovered by Einstein in 1915; direct detection of gravitational waves has been achieved only recently, almost a hundred years after the prediction, by the LIGO experiment [15]. And a static point source produces (under suitable conditions) a Newtonian potential that falls off as  $1/r$  just as its scalar counterpart does.

Before we leave the comfortable familiarity of four dimensional space-time it is useful to briefly discuss a variant on the scalar wave equation called the Klein-Gordon model or massive scalar field theory. In this model the scalar field obeys

$$\square^2 \phi + \mu^2 \phi = s. \quad (5)$$

Here the “mass parameter”  $\mu$  has units of inverse length. We may regard the conventional wave eq (2) as the massless limit,  $\mu \rightarrow 0$ , of the Klein Gordon model. In the absence of sources, plane waves,  $f(\mathbf{r}; \mathbf{k}) \exp(i\omega t)$ , are still solutions, with  $f$  given by eq (2), but the frequency and wave-vector are related by the dispersion relation  $\omega = \sqrt{k^2 + \mu^2}$ .

Parenthetically we note that in quantum field theory the Klein-Gordon model describes massive bosons. This can be motivated by multiplying both sides of the dispersion relation by  $\hbar$  and recalling that  $\hbar\omega$  is the energy

and  $\hbar k$  the momentum by Planck's formula and the de Broglie relation respectively. Then comparing the dispersion relation to the energy-momentum relation of a particle of mass  $m$  in special relativity reveals that the Klein-Gordon model describes a particle of mass  $m = \mu\hbar/c$ , where we have momentarily restored the factors of  $c$ .

Now let us solve the Klein-Gordon model for a static source. In this case the field  $\phi$  will also be time independent. Again it is sufficient to analyze a point source  $s = \lambda\delta(\mathbf{r})$ ; more general static solutions can be developed by superposition. Taking the Fourier transform with respect to the spatial dependence in eq (5) yields  $(k^2 + \mu^2)\tilde{\phi} = \lambda$ . Inverting the Fourier transform yields the Yukawa potential

$$\phi(\mathbf{r}) = \lambda \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{k^2 + \mu^2} \exp(i\mathbf{k} \cdot \mathbf{r}) = \frac{\lambda}{4\pi r} \exp(-\mu r). \quad (6)$$

At short distances,  $r \ll \mu^{-1}$ , the Yukawa potential reproduces the massless  $1/r$  behavior but at long distances,  $r \gg \mu^{-1}$ , the Yukawa potential vanishes exponentially.

It is a good exercise for the reader to verify eq (6). Again the Fourier integral is most easily evaluated by working in spherical polar co-ordinates and integrating over the angular variables first. The radial integral can be performed by making use of the definite integral

$$\int_0^\infty dk \frac{k \sin kr}{k^2 + \mu^2} = \frac{\pi}{2} \exp(-\mu r). \quad (7)$$

Eq (7) can be obtained by contour integration or by more elementary means [26].

### III. ADD MODEL: THE INCREDIBLE BULK

*The hierarchy problem:* In George Gamow's fable [16], the initials of Mr C.G.H. Tompkins are an allusion to the three fundamental constants of nature,  $c, G$  and  $\hbar$ . From these constants one can construct only one expression with units of energy, the Planck energy, given by  $E_P = \sqrt{\hbar c^5/G}$ . Putting in numbers it emerges that  $E_P = 1.2 \times 10^{28}$  eV. This is far above 1 TeV, the scale of electroweak theory. The discrepancy between the fundamental Planck scale and the electroweak scale is called the hierarchy problem. Especially within quantum field theory it is difficult to understand how the two energy scales could be so far apart. Note that as  $G \rightarrow 0$ , the Planck scale  $\rightarrow \infty$ . Thus the hierarchy problem would worsen if gravity were weaker. Hence we may regard the hierarchy problem as a measure of the weakness of gravity compared to other forces.

*Brane and Bulk.* In the simplest version of the ADD model space has a fourth dimension. The position of a point is specified by four Cartesian co-ordinates  $(x, y, z, w)$ . There is an additional twist that the fourth dimension is rolled up with a circumference  $L$ . More precisely, the extra dimension is periodic with a period  $L$ .

Thus  $(x, y, z, w)$  and  $(x, y, z, w + L)$  correspond to the same point. In this model space-time is five-dimensional. An event is specified by the co-ordinates  $(t, x, y, z, w)$ . The space-time interval between two nearby events located at  $(t, x, y, z, w)$  and  $(t + dt, x + dx, y + dy, z + dz, w + dw)$  is given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - dw^2. \quad (8)$$

The ADD space-time is therefore flat. The four dimensional section of ADD space-time corresponding to  $w = 0$  is called the brane. The co-ordinates of the brane lie in the range  $-\infty < t, x, y, z < \infty$ . Standard model particles and fields are assumed to be confined to live on the brane. Gravity however propagates throughout the five dimensional space-time which is called the Bulk.

*Zero-modes and Kaluza-Klein modes.* The scalar wave equation in ADD space-time has the form

$$\square^2 \phi = \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial w^2} \right) \phi = s \quad (9)$$

where  $\phi$  is the scalar field and  $s$  is its source. In the absence of sources the wave equation has plane wave solutions  $\phi = f(\mathbf{r}, w; \mathbf{k}, \kappa_n) \exp(i\omega t)$  where the mode function

$$f(\mathbf{r}, w; \mathbf{k}, \kappa_n) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{L}} \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(i\kappa_n w) \quad (10)$$

and the frequency  $\omega$  obeys the dispersion relation  $\omega = \sqrt{k^2 + \kappa_n^2}$ . In order to ensure the periodicity  $f(\mathbf{r}, w + L; \mathbf{k}, \kappa_n) = f(\mathbf{r}, w; \mathbf{k}, \kappa_n)$  the extra dimensional component of the wave-vector  $\kappa_n$  obeys the quantization

$$\kappa_n = \frac{2\pi n}{L} \quad (11)$$

where  $n = \dots, -1, 0, 1, 2, \dots$  is an integer. Waves with  $n = 0$  are called zero-modes. Zero modes behave essentially like waves in four dimensional space-time. They are labelled by a wave vector  $\mathbf{k}$  that only has components along the brane and they have the massless dispersion  $\omega = k$  characteristic of waves in four dimensional space-time. Modes with non-zero  $n$  are called Kaluza-Klein modes. Kaluza-Klein modes have an  $\omega$ - $\mathbf{k}$  dispersion corresponding to the massive wave equation with a mass parameter  $\mu \rightarrow \kappa_n$ .

*Point source.* Next let us examine the solution to the wave equation (10) for a static source that is confined to the brane. As before we consider a point source  $s = \lambda\delta(\mathbf{r})\delta(w)$ ; more complicated distributions can be treated by superposition. Evidently the resulting field  $\phi$  will be time independent and has the Fourier expansion

$$\phi(\mathbf{r}, w) = \sum_{n=-\infty}^{\infty} \int d\mathbf{k} \tilde{\phi}(\mathbf{k}, \kappa_n) f(\mathbf{r}, w; \mathbf{k}, \kappa_n) \quad (12)$$

where the Fourier amplitudes  $\tilde{\phi}(\mathbf{k}, \kappa_n)$  are given by

$$\tilde{\phi}(\mathbf{k}, \kappa_n) = \int d\mathbf{r} \int_0^L dw \phi(\mathbf{r}, w) f^*(\mathbf{r}, w; \mathbf{k}, \kappa_n). \quad (13)$$

In Fourier space the wave eq (10) takes the form  $(k^2 + \kappa_n^2)\tilde{\phi} = \lambda/\sqrt{L}(2\pi)^{3/2}$  and hence

$$\phi(\mathbf{r}, w) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\lambda}{k^2 + \kappa_n^2} \exp(i\mathbf{k}\cdot\mathbf{r}) \exp(i\kappa_n w). \quad (14)$$

The  $\mathbf{k}$  integral can be performed by comparison to eq (6). Hereafter let us consider only the value of the field on the brane and set  $w = 0$ . We obtain

$$\phi(\mathbf{r}, 0) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{\lambda}{4\pi r} \exp\left(-\frac{2\pi|n|r}{L}\right) \quad (15)$$

Eq (15) is the field of a point source that is located on the brane at the origin at a distance  $r$  from the source. The sum over modes,  $n$ , can be performed exactly but it is more illuminating to consider limiting cases. For  $r \gg L$  only the zero mode ( $n = 0$ ) contributes to the field—the contribution of the Kaluza-Klein modes is exponentially suppressed. We obtain

$$\phi(\mathbf{r}, 0) \approx \frac{\lambda}{L} \frac{1}{4\pi r} \quad (16)$$

At distances that are long compared to the size of the extra dimension we obtain the  $1/r$  behavior characteristic of four space-time dimensions; the extra dimension is essentially invisible. On the other hand for  $r \ll L$  all modes contribute and we obtain

$$\phi(\mathbf{r}, 0) \approx \frac{\lambda}{4\pi^2} \frac{1}{r^2}. \quad (17)$$

At distances that are short compared to the extra dimension we obtain the  $1/r^2$  behavior characteristic of five space-time dimensions.

*Hierarchy revisited.* Standard model fields are confined to the brane in the ADD model. Only gravity propagates through the bulk. Above we have analyzed a scalar field that propagates in the bulk as a proxy for the gravitational field. Scalar fields are simpler than gravitational fields. Moreover it turns out that—just as in Minkowski space-time—scalar fields are a good cartoon for gravitational fields and provide valuable insights. Suppose we were interested in the gravitational field of a point mass  $m$  that lives on the brane. At short distances we expect that the Newtonian potential will have the five dimensional form  $G_5 m/r^2$  where  $G_5$  is the gravitational interaction. On the other hand at long distances we expect the Newtonian potential would have the four dimensional form  $Gm/r^2$  where  $G$  is the familiar Newtonian constant of gravity. Comparing these forms to eqs (16) and (17) we infer that  $\lambda = 4\pi^2 G_5 m = 4\pi GLm$ . Hence we arrive at the important result

$$G_5 = \frac{GL}{\pi}. \quad (18)$$

Eq (18) lies at the heart of the ADD solution to the hierarchy problem. The idea is that the five dimensional

Planck scale should be determined by  $c, \hbar$  and  $G_5$ . The dimensions of  $G_5$  have an extra factor of length compared to  $G$ . Simple dimensional analysis shows that the five dimensional Planck scale should be given by  $E_{P5} = (\hbar^2 c^6 / G_5)^{1/3}$ . In this expression we replace  $G_5 \rightarrow GL$ , in accordance with eq (18), and impose the requirement that  $E_{P5} \sim 1\text{TeV}$  (the electroweak scale) to obtain  $L \sim 10^{13}\text{m}$ . Thus the hierarchy problem is solved if we posit the existence of an extra dimension of this enormous size.

Similarly if there were two large extra dimensions we would have  $G_6 \sim GL^2$ . On dimensional grounds, the Planck scale would be given by  $(\hbar^3 c^7 / G_6)^{1/4}$ . Replacing  $G_6 \rightarrow GL^2$  and imposing the requirement that  $E_P \sim 1\text{TeV}$  yields  $L \sim 1\text{mm}$ . Thus the six dimensional ADD model solves the hierarchy problem if we posit the existence of two extra dimensions on the millimeter scale (which is still enormous compared to the Planck scale).

Experiments do rule out the existence of a single extra dimension of order  $10^{13}\text{m}$ . Most directly the gravitational potential is observed to vary as  $1/r$  on much shorter distance scales. However millimeter scale extra dimensions could not be excluded by experiment at the time of the ADD paper. Thus the possibility arose that large extra dimensions might exist and explain the hierarchy problem.

#### IV. RANDALL-SUNDRUM MODEL: WARPED EXTRA DIMENSIONS

*Warp factor.* In the simplest version of the Randall-Sundrum model space has a fourth dimension. An event in space-time is specified by the five co-ordinates  $(t, x, y, z, w)$ . The co-ordinate  $w$  corresponds to the extra dimension and is assumed to lie in the range  $0 \leq w < \infty$ . The other co-ordinates have the usual range  $-\infty < t, x, y, z < \infty$ . The four dimensional section corresponding to  $w = 0$  is called the brane. The complete five dimensional space-time is the bulk. The space-time interval between neighboring events located at  $(t, x, y, z, w)$  and  $(t + dt, x + dx, y + dy, z + dz, w + dw)$  is given by [27]

$$ds^2 = e^{-2\gamma w} dt^2 - e^{-2\gamma w} (dx^2 + dy^2 + dz^2) - dw^2. \quad (19)$$

$\gamma$  is a parameter with units of inverse length. Thus the Randall Sundrum space-time is warped. For fixed  $w$  the space-time is essentially flat Minkowski space-time but with an overall  $w$  dependent “warp factor”  $e^{-2\gamma w}$ . There is no system of co-ordinates in which the space-time interval has the flat form eq (8) globally. Readers without a background in general relativity will therefore find it useful to read Appendix A before proceeding with the remainder of this section.

*Conformal co-ordinates.* Although in principle we could continue to work with the co-ordinates  $(t, x, y, z, w)$  that were used by Randall and Sundrum, in practice it is more convenient to work with a different system

called conformal co-ordinates. Instead of  $w$  we use a co-ordinate  $\zeta$ . The two are related via  $\zeta = e^{\gamma w}/\gamma$ . Evidently  $\gamma^{-1} \leq \zeta < \infty$  and  $\zeta = \gamma^{-1}$  corresponds to the brane. In terms of conformal co-ordinates the space-time interval is given by

$$ds^2 = \frac{1}{\gamma^2 \zeta^2} (dt^2 - dx^2 - dy^2 - dz^2 - d\zeta^2) \quad (20)$$

as the diligent reader should verify. Eq (20) reveals that the Randall Sundrum space-time has the same interval as the flat ADD space-time eq (8) up to an overall factor. The two space-times are said to be conformally equivalent.

*Scalar waves.* In conformal co-ordinates we see that all the scale factors are equal:  $h_t = h_x = h_y = h_z = h_\zeta = 1/\gamma\zeta$  and the invariant measure  $h = 1/(\gamma\zeta)^5$ . In conformal co-ordinates the scalar wave equation has the form

$$-\frac{\partial^2}{\partial \zeta^2} \phi + \frac{3}{\zeta} \frac{\partial}{\partial \zeta} \phi - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi + \frac{\partial^2}{\partial t^2} \phi = \gamma^2 \zeta^2 s. \quad (21)$$

Here  $\phi$  is the scalar field and  $s$  is the source. The reader should verify that eq (21) follows from the space-time interval eq (20) following the methods of appendix A. In order to fully specify the problem, in addition to the wave eq (21), we also need to specify the boundary condition satisfied by the scalar field on the brane. We will specify the boundary conditions momentarily. The form of eq (21) suggest that the solutions in the absence of sources will have the form of plane waves modulated by a  $\zeta$  dependent factor  $v(\zeta)$ , namely,  $\phi = v(\zeta) \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(i\omega t)$ . Substitution of this ansatz in eq (21) reveals that the modulation factor obeys

$$\frac{d^2 v}{d\zeta^2} - \frac{3}{\zeta} \frac{dv}{d\zeta} + (\omega^2 - k^2)v = 0. \quad (22)$$

We will see below that  $\omega^2 \geq k^2$ ; hence it is convenient to write  $\omega^2 - k^2$  as  $\mu^2$  in eq (22).

*Schrödinger analogy.* Following [5] we can gain much insight into the solutions to eq (22) by transforming to the independent variable  $\psi$  defined by  $v = \zeta^{3/2} \psi$ . Eq (22) then has the form

$$-\frac{1}{2} \frac{d^2}{d\zeta^2} \psi + \frac{15}{8} \frac{1}{\zeta^2} \psi = \frac{\mu^2}{2} \psi. \quad (23)$$

Eq (23) coincides with the time-independent Schrödinger equation for a particle that is confined to the half line  $\zeta > \gamma^{-1}$  and that experiences a potential barrier  $15/8\zeta^2$ . In this analogy the “energy” of the particle is  $\mu^2/2$ . Thus we can draw upon our intuition from non-relativistic quantum mechanics in order to deduce the solutions to eq (23). The form of the potential suggests that there should be a continuum of states with  $\mu^2 > 0$ . These states constitute the Kaluza-Klein modes of the Randall-Sundrum model. In addition there is a single bound state which

corresponds to the zero-mode of the Randall-Sundrum model.

*Zero mode.* It may seem surprising that eq (23) has a bound state solution since the potential is purely repulsive. However we still have the freedom to choose the boundary condition we apply to the “wave-function”  $\psi$  at the location of the brane  $\zeta = \gamma^{-1}$ . The boundary condition will have the form  $\psi(\gamma^{-1}) = \lambda \psi'(\gamma^{-1})$  where  $\lambda$  is a parameter that we have freedom to specify. The boundary condition effectively functions like a contact interaction which can be repulsive or attractive [28]. By choosing the right boundary condition we can ensure that there will be a bound state with  $\mu = 0$ . Indeed if we set  $\mu = 0$  it is easy to see that eq (23) has power law solutions of the form  $\psi = \zeta^\beta$  with  $\beta = -\frac{3}{2}$  or  $\beta = \frac{5}{2}$ . We discard the latter on the grounds that it is not normalizable. Normalizing the former we conclude that eq (23) has the bound state solution with  $\mu = 0$  given by

$$\psi_0 = \frac{\sqrt{2}}{\gamma} \zeta^{-3/2} \quad (24)$$

that satisfies the normalization condition  $\int_{\gamma^{-1}}^{\infty} d\zeta |\psi_0|^2 = 1$ . The solution (24) satisfies the boundary condition  $\psi(\gamma^{-1}) = \lambda \psi'(\gamma^{-1})$  with  $\lambda = -\frac{2}{3}\gamma^{-1}$  and hence this is the boundary condition we will impose on the scalar field at the brane hereafter [29].

*Kaluza-Klein modes.* Determination of the positive energy continuum of solutions to eq (23) is a straightforward exercise in one dimensional quantum mechanics. We relegate the details to appendix B and only quote the results here. The solutions have the form

$$\psi(\zeta; \mu) = \sqrt{\zeta} [\alpha(\mu) J_2(\mu\zeta) + \beta(\mu) Y_2(\mu\zeta)]. \quad (25)$$

Here  $J_2$  and  $Y_2$  denote the Bessel and Neumann functions of second order. Eq (25) is a solution to eq (23) for arbitrary coefficients  $\alpha$  and  $\beta$ . The boundary condition on the brane at  $\zeta = \gamma^{-1}$  fixes the ratio of  $\alpha$  to  $\beta$ . We find

$$\alpha(\mu) = AY_1\left(\frac{\mu}{\gamma}\right) \quad \text{and} \quad \beta(\mu) = -AJ_1\left(\frac{\mu}{\gamma}\right). \quad (26)$$

We choose the overall constant  $A$  by imposing the condition that  $\alpha^2 + \beta^2 = \mu$ . As shown in appendix B this choice ensures that the continuum solutions satisfy the normalization condition

$$\int_{\gamma^{-1}}^{\infty} d\zeta \psi^*(\zeta; \mu) \psi(\zeta; \mu') = \delta(\mu - \mu'). \quad (27)$$

Recall that solutions to the Schrödinger equation constitute a complete set. For later use it is worth noting that the continuum solutions derived here together with the zero mode eq (24) satisfy the completeness relation

$$\psi_0^*(\zeta) \psi_0(\zeta') + \int_0^\infty d\mu \psi^*(\zeta; \mu) \psi(\zeta'; \mu) = \delta(\zeta - \zeta'). \quad (28)$$

*Enumeration of modes.* In summary the scalar wave equation in Randall Sundrum space-time has two kinds of solutions in the absence of sources. The zero-mode solutions are labelled by their three dimensional wave-vector  $\mathbf{k}$  and have the form  $g(\zeta, \mathbf{r}; \mathbf{k}) \exp(i\omega t)$ . The mode function  $g$  is given by

$$g(\zeta, \mathbf{r}; \mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \zeta^{3/2} \psi_0(\zeta) \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (29)$$

where  $\psi_0$  is given by eq (24). Zero modes have the dispersion relation  $\omega = k$  characteristic of waves in four space-time dimensions. The second class of solutions are the Kaluza-Klein modes labelled by their three dimensional wave-vector  $\mathbf{k}$  and their mass parameter  $\mu$ . The Kaluza-Klein modes have the form  $f(\zeta, \mathbf{r}; \mathbf{k}, \mu) \exp(i\omega t)$ . The mode function  $f$  is given by

$$f(\zeta, \mathbf{r}; \mathbf{k}, \mu) = \frac{1}{(2\pi)^{3/2}} \zeta^{3/2} \psi(\zeta; \mu) \exp(i\mathbf{k} \cdot \mathbf{r}). \quad (30)$$

$\psi(\zeta; \mu)$  is given by eq (25). The Kaluza-Klein modes have the  $\omega - \mathbf{k}$  dispersion relation  $\omega = \sqrt{k^2 + \mu^2}$  corresponding to the massive wave equation with mass parameter  $\mu$ . In contrast to the ADD model, which had a discrete set of Kaluza-Klein modes with quantized mass parameters, the Randall-Sundrum model has a continuum of Kaluza-Klein modes and the mass parameter can take on any positive real value.

*Point source.* Finally let us consider the solution to a static point source confined to the brane. Evidently the resulting field  $\phi$  will also be time independent. We wish to show, following [5], that the field produced by such a source falls off inversely with distance at distances that are long compared with  $\gamma^{-1}$ . Hence in the Randall Sundrum model too the extra dimension is hidden on long length scales.

For a point source localized on the brane,  $s = \lambda \delta(\mathbf{r}) \delta(\zeta - \gamma^{-1})$ . Taking into account that the solution  $\phi$  will be time-independent, eq (21) has the form

$$-\nabla_{\text{RS}}^2 \phi = \lambda \delta(\mathbf{r}) \delta(\zeta - \gamma^{-1}) \quad (31)$$

where

$$\nabla_{\text{RS}}^2 = \left( \frac{\partial^2}{\partial \zeta^2} - \frac{3}{\zeta} \frac{\partial}{\partial \zeta} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right). \quad (32)$$

In order to solve eq (31) we write the solution  $\phi(\mathbf{r}, \zeta)$  as a superposition of zero-mode and Kaluza-Klein mode functions. Thus

$$\begin{aligned} \phi(\zeta, \mathbf{r}) &= \int d\mathbf{k} \tilde{\phi}_0(\mathbf{k}) g(\zeta, \mathbf{r}; \mathbf{k}) \\ &+ \int d\mathbf{k} \int_0^\infty d\mu \tilde{\phi}(\mathbf{k}, \mu) f(\zeta, \mathbf{r}; \mathbf{k}, \mu). \end{aligned} \quad (33)$$

The amplitudes  $\tilde{\phi}_0(\mathbf{k})$  and  $\tilde{\phi}(\mathbf{k}, \mu)$  are to be determined.

It is now helpful to note that

$$\begin{aligned} -\nabla_{\text{RS}}^2 g(\zeta, \mathbf{r}; \mathbf{k}) &= k^2 g, \\ -\nabla_{\text{RS}}^2 f(\zeta, \mathbf{r}; \mathbf{k}, \mu) &= (k^2 + \mu^2) f. \end{aligned} \quad (34)$$

Eq (34) follows because  $g \exp(i\omega t)$  and  $f \exp(i\omega t)$  obey the scalar wave eq (21), and from the dispersion relations for the zero and Kaluza-Klein modes respectively. Using eqs (33) and (34) we can write the left hand side of eq (31) as

$$\begin{aligned} -\nabla_{\text{RS}}^2 \phi(\zeta, \mathbf{r}) &= \int d\mathbf{k} k^2 \tilde{\phi}_0(\mathbf{k}) g(\zeta, \mathbf{r}; \mathbf{k}) \\ &+ \int d\mathbf{k} \int_0^\infty d\mu (k^2 + \mu^2) \tilde{\phi}(\mathbf{k}, \mu) f(\zeta, \mathbf{r}; \mathbf{k}, \mu). \end{aligned} \quad (35)$$

On the other hand we can obtain the right hand side of eq (31) by substituting  $\zeta' \rightarrow \gamma^{-1}$  and  $\mathbf{r}' \rightarrow 0$  in the identity [30]

$$\begin{aligned} \zeta'^3 \delta(\zeta - \zeta') \delta(\mathbf{r} - \mathbf{r}') &= \int d\mathbf{k} g^*(\zeta', \mathbf{r}'; \mathbf{k}) g(\zeta, \mathbf{r}; \mathbf{k}) \\ &+ \int d\mathbf{k} \int_0^\infty d\mu f^*(\zeta', \mathbf{r}'; \mathbf{k}, \mu) f(\zeta, \mathbf{r}; \mathbf{k}, \mu). \end{aligned} \quad (36)$$

Making these substitutions and equating to eq (35) yields

$$\begin{aligned} \tilde{\phi}_0(\mathbf{k}) &= \lambda \frac{\gamma^3}{k^2} g^* \left( \frac{1}{\gamma}, \mathbf{0}; \mathbf{k} \right), \\ \tilde{\phi}(\mathbf{k}, \mu) &= \lambda \frac{\gamma^3}{k^2 + \mu^2} f^* \left( \frac{1}{\gamma}, \mathbf{0}; \mathbf{k}, \mu \right). \end{aligned} \quad (37)$$

This completes the determination of the amplitudes  $\tilde{\phi}_0$  and  $\tilde{\phi}$ .

Substituting eq (37) in eq (33) and making use of the explicit forms of  $g$  and  $f$  given in eqs (29) and (30) we obtain

$$\begin{aligned} \phi(\zeta, \mathbf{r}) &= 2\lambda\gamma \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{k^2} \exp(i\mathbf{k} \cdot \mathbf{r}) \\ &+ \lambda\gamma\zeta^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int_0^\infty d\mu \frac{1}{k^2 + \mu^2} \exp(i\mathbf{k} \cdot \mathbf{r}) \\ &\times \left[ \alpha(\mu) J_2 \left( \frac{\mu}{\gamma} \right) + \beta(\mu) Y_2 \left( \frac{\mu}{\gamma} \right) \right]^2. \end{aligned} \quad (38)$$

The first term corresponds to the zero-mode contribution the field; the second to the Kaluza-Klein modes. Performing the integrals over  $\mathbf{k}$ , which are the same as in eqs (4) and (6), yields

$$\begin{aligned} \phi(\zeta, \mathbf{r}) &= \frac{\lambda\gamma}{2\pi r} + \frac{\lambda\gamma\zeta^2}{4\pi r} \int_0^\infty d\mu \exp(-\mu r) \\ &\times \left[ \alpha(\mu) J_2 \left( \frac{\mu}{\gamma} \right) + \beta(\mu) Y_2 \left( \frac{\mu}{\gamma} \right) \right]^2. \end{aligned} \quad (39)$$

Eq (39) is the exact expression for the field due to a point source located at the brane. It is instructive to simplify it in the limit of long distance  $r \gg \gamma^{-1}$ . In this limit, due to the decaying exponential it is acceptable to approximate the remaining terms in the integrand by their small  $\mu$  limit. The expansion is tedious by hand but

Mathematica readily finds that the square of the term in square brackets in eq (39) is approximately equal to  $\mu$  in the small  $\mu$  limit. Making this approximation and evaluating the integral over  $\mu$  we finally obtain

$$\phi(\gamma^{-1}, \mathbf{r}) \approx \frac{\lambda\gamma}{2\pi r} \left(1 + \frac{1}{2\gamma^2 r^2} + \dots\right) \quad (40)$$

for  $r \gg \gamma^{-1}$ . We have also set  $\zeta \rightarrow \gamma^{-1}$  since we are primarily interested in the field on the brane. Thus as advertised at long distances a point source couples primarily to the zero mode and produces a field that varies at  $1/r$ . Thus the extra dimension is hidden. The leading correction due to coupling to the Kaluza Klein modes varies as  $1/r^3$  rather than exponentially as in the ADD model.

In principle we can work out the small  $r$  behavior also from eq (39). In practice it is simpler to recall that locally any space or space-time looks flat. The expression for the field of a point source in a flat four dimensional space can be obtained by taking the  $L \rightarrow \infty$  limit of eq (15) which happens to be the same as the  $r \ll L$  result, eq (17). Thus we conclude that

$$\phi(\gamma^{-1}, \mathbf{r}) \approx \frac{\lambda}{4\pi^2 r^2} \quad (41)$$

for  $r \ll \gamma^{-1}$  in the Randall Sundrum model.

*Implications for Gravity.* Randall and Sundrum assumed that standard model particles and fields were confined to the brane. Only gravity was assumed to propagate through the bulk. Here we have analyzed a bulk scalar field as a proxy for the gravitational field. Scalar fields are simpler to analyze but have useful parallels to gravitational fields. For example gravitational waves also have a discrete zero mode and a continuum of Kaluza-Klein modes; indeed the same Schrödinger analogy and the same Bessel functions arise in both cases.

Suppose now that we are interested in the gravitational field of a point mass  $m$  located on the brane. At short distances we expect its Newtonian gravitational potential will have the five dimensional form  $G_5 m/r^2$  where  $G_5$  is the gravitational coupling constant. On the other hand at long distances we expect the Newtonian potential to have the four dimensional form  $Gm/r$  where  $G$  is the familiar Newtonian constant of gravity. Comparing these forms to eqs (40) and (41) we infer that  $\lambda = 4\pi^2 G_5 m = 2\pi Gm/\gamma$  which implies

$$G_5 = \frac{G}{2\pi\gamma}. \quad (42)$$

Eq (42) agrees with eq (7) of ref [5] and is analogous to eq (18) for the ADD model and is the key to resolving the hierarchy problem within this model.

*RS1 Model: large hierarchy, small extra dimension.* The model we have analyzed thus far was introduced in [5] and is sometimes called the RS2 model. In their earlier paper [4] Randall and Sundrum introduced a slightly different model called RS1 which provides an especially

elegant resolution of the hierarchy problem. In this model the geometry of space-time is still given by eq (19) but the co-ordinate  $w$  only extends over the range  $0 \leq w \leq \ell$ . In addition to the brane at  $w = 0$ , a second brane is located at  $w = \ell$ . We shall refer to these branes as the left brane and the right brane respectively. In conformal co-ordinates the left brane is located at  $\zeta = \gamma^{-1}$  and the right brane at  $\zeta = \gamma^{-1} \exp(\gamma\ell)$ . In RS1 the zero mode (29) remains a solution that satisfies the boundary condition at both branes. The Kaluza-Klein modes become quantized: it is only for particular values of  $\mu$  that the solutions (30) obey the boundary conditions at both branes. The lowest allowed value of  $\mu$  is of order  $\gamma \exp(-\gamma\ell)$  and the spacing between successive allowed values of  $\mu$  is also of this order (see problem 6 in appendix C for a derivation). In RS1 we choose  $\gamma^{-1}$  and  $\ell$  to both be of order the Planck length, but if we choose the product  $\gamma\ell \approx 35-40$ , then the lowest allowed  $\mu$  is sixteen orders of magnitude smaller, and is of the weak scale. Thus a large hierarchy can emerge in RS1 due to the exponential warp factor even though the basic parameters of the model are all of Planck scale.

To better understand the hierarchy as it applies to the standard model consider a scalar field  $\xi$  that is confined to the right brane. This field is meant as a proxy for standard model fields which are confined to the right brane in RS1. We take the field to be governed by the Klein-Gordon equation  $\square_R^2 \xi + m^2 \xi = 0$ , where  $\square_R^2$  is the d'Alembertian appropriate for the right brane, and  $m$  is the mass parameter of the field, which is taken to be Planck scale, consistent with the rule that all parameters in RS1 are Planck scale. This equation has the form (see problem 7, appendix C)

$$e^{2\gamma\ell} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \xi + m^2 \xi = 0. \quad (43)$$

By introducing the rescaled field  $\bar{\xi} = \xi \exp(2\gamma\ell)$  we can bring eq (43) to the form

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \bar{\xi} + m^2 e^{-2\gamma\ell} \bar{\xi} = 0. \quad (44)$$

Now that the derivative terms have been brought to the canonical form for a Klein-Gordon equation we can apply the standard interpretation that eq (44) describes scalar particles of mass  $(m\hbar/c)e^{-\gamma\ell}$ . If we take  $(m\hbar/c)$  to be of order a Planck mass then by choosing  $\gamma\ell \approx 35-40$  we can arrange for the standard model particles to have a mass of order 1TeV, the weak scale. It is a good exercise to verify that if the Klein-Gordon field had lived on the left brane it would have described a particle with a mass of order the Planck mass. Thus it is important that standard model particles and fields be localized on the right brane, not the left, in RS1.

*Comparison to ADD: colliders and torsion pendulums.* Both RS1 and ADD predict that the gravitational force will depart from the Newton's inverse square law at small distances. However for RS1 this happens at unobservable short distances whereas for the ADD model this

could happen at close to millimeter scales that are accessible to table top experiments. Experiments with torsion pendulums that have verified Newton's law down to a tenth of a millimeter represent some of the tightest constraints on large extra dimensions [17]. Taking into account quantum mechanics, in both models the zero mode corresponds to a massless particle—the graviton. Each Kaluza-Klein mode can be thought of as a separate particle with a mass equal to  $\mu\hbar/c$  where  $\mu$  is the mass parameter of the mode. Thus both models predict an infinite tower of new particles. The mass of the lightest Kaluza-Klein particle in the ADD model is  $10^{-4}$  eV, assuming two extra dimensions of millimeter scale (since  $\mu \sim 1/L$  for the ADD model). This is also the scale of the separation in mass between successive Kaluza-Klein particles. In RS1 the lightest Kaluza-Klein particle has a much larger mass of order 1 TeV (since  $\mu \sim \gamma \exp^{-\gamma\ell}$ ) and that is also the order of the spacing between successive Kaluza-Klein particle masses. Furthermore, although this is not evident from our presentation, the Kaluza-Klein particles in RS1 interact strongly with standard model particles on the TeV scale, whereas the interaction with the corresponding ADD particles is only of gravitational strength. Thus the Kaluza-Klein modes in RS1 would be manifested in collider experiments as well defined particles with masses on the TeV scale. The Kaluza-Klein modes of the ADD model would not be individually observable, but because they are so numerous, their production would be manifested as missing energy in collisions [7]. Astrophysical observations are also sensitive to the production of Kaluza-Klein particles in the ADD model. For example if the production becomes too copious it would put supernovae models in conflict with data [7].

*Strings and Stability.* In an important further development ref [18] showed that the inter-brane separation  $\ell$  in RS1 could be stabilized by postulating the existence of a massive scalar field that propagated in the bulk. Incorporating this stabilization, the RS1 model becomes an elegant and fully consistent explanation of the hierarchy problem. By contrast the ADD model suffers from a hierarchy problem of its own [19] (see problem 2, appendix C). The Randall-Sundrum model has profound connections to string theory, notably to a model studied in context of  $M$ -theory by Horava and Witten [20]. The presumption is that the Randall-Sundrum model is an effective field theory that applies below the Planck scale as an approximation to a more fundamental theory that would be needed to describe physics above the Planck scale.

## V. DGP MODEL: AN ALTERNATIVE TO DARK ENERGY

*Dielectric analogy.* As a preliminary it is instructive to discuss a solid-state physics problem, that of screening by two dimensional dielectrics (relevant for example to

dielectric materials related to graphene) [21]. Consider a planar dielectric sheet with an external point charge  $q$  placed at the origin. As a result the bound charges in the dielectric sheet are polarized. The polarization vector lies in the plane of the sheet and is given by

$$P_x = -\ell \frac{\partial}{\partial x} \phi \delta(z) \quad \text{and} \quad P_y = -\ell \frac{\partial}{\partial y} \phi \delta(z). \quad (45)$$

Here  $\ell$  denotes the polarizability of the sheet, which is assumed to lie in the  $x$ - $y$  plane, and  $\phi$  is the electrostatic scalar potential. The resulting bound charge density is then given by the negative divergence of the polarization. Thus

$$\rho_{\text{pol}} = \ell \left( \frac{\partial^2}{\partial x^2} \phi + \frac{\partial^2}{\partial y^2} \phi \right) \delta(z). \quad (46)$$

By Gauss's law  $\nabla \cdot \mathbf{E} = q\delta(\mathbf{r}) + \rho_{\text{pol}}$ . Thus Poisson's equation takes the form

$$-\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \phi - \ell \delta(z) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \phi = q\delta(\mathbf{r}). \quad (47)$$

Eq (47) is easily solved using Fourier analysis. For the sake of brevity we omit the details of the solution and only quote the results. It is found that the electric field in the  $x$ - $y$  plane is that of a bare point charge  $q$  at distances that are large compared to  $\ell$ . Thus the dielectric sheet produces no screening at long distances. At short distances however the electric field appears to be that of an infinite line of charge with a charge density  $q/\ell$ . Thus at long distances the electric field varies with distance as  $1/r^2$  appropriate for a point charge in three dimensions; but at short distances the electric field varies with distance as  $1/r$  appropriate for a point charge in two dimensions.

*The model.* In the DGP model there is an extra dimension of space. An event is specified by the co-ordinates  $(t, x, y, z, w)$ . The space-time interval between nearby events is given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - dw^2. \quad (48)$$

All space-time co-ordinates have the same infinite range  $-\infty < t, x, y, z, w < \infty$ . Thus DGP space-time is both flat and possesses an infinite extra dimension. The four dimensional section  $w = 0$  is the brane; standard model particles and fields are confined to it. Gravity however propagates throughout the bulk. Since the equations that govern gravitational fields are complicated we instead consider a scalar field that propagates through the bulk. The key idea in the DGP model is that the brane is analogous to the dielectric sheet discussed above. Thus a scalar field that propagates in the bulk is assumed to be governed by the wave equation

$$\begin{aligned} & \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial w^2} \right) \phi \\ & + \ell \delta(w) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \phi = s(\mathbf{r}, w, t). \end{aligned} \quad (49)$$



Eq (49) is a natural generalization of eq (47) that incorporates dynamics by including time derivatives and that also incorporates an extra dimension of space.  $\ell$  is a parameter with units of length that we shall call the screening length.

*Point source.* We now consider the field produced by a static point source localized on the brane. Thus the source is  $s \rightarrow \lambda \delta(\mathbf{r}) \delta(w)$ . The resulting field will also be static and we are primarily interested in the field on the brane,  $\phi(\mathbf{r}, w \rightarrow 0)$ . It is found that at short distances ( $r \ll \ell$ )

$$\phi(\mathbf{r}, 0) \approx \frac{1}{4\pi} \frac{\lambda}{\ell} \frac{1}{r} \quad (50)$$

whereas at long distances ( $r \gg \ell$ )

$$\phi(\mathbf{r}, 0) \approx \frac{\lambda}{4\pi^2} \frac{1}{r^2}. \quad (51)$$

Remarkably we obtain behavior appropriate to four space-time dimensions at short distances and to five space-time dimensions at long distances. In contrast to the ADD and Randall-Sundrum models, here the extra dimension reveals itself at long distances and remains hidden at short. Eq (50) and (51) can be derived by solving eq (49) using Fourier analysis. The Fourier solution is straightforward and is relegated to the problems. Intuitively we can understand the results as follows. Evidently  $\phi(\mathbf{r}, 0)$  must depend on  $r$ ,  $\ell$  and  $\lambda$ . On dimensional grounds we expect  $\phi(\mathbf{r}, 0) = (\lambda/r^2)f(\ell/r)$  where  $f$  is an unknown function. This form suggests that to obtain the  $\ell \ll r$  limit it is sufficient to consider the  $\ell \rightarrow 0$  limit of eq (49). In that limit eq (49) reduces to the electrostatics of a point source in four space dimensions. Hence  $\phi$  has the same form as in the small  $r$  limits of the ADD and Randall-Sundrum models; compare eqs (17), (41) and (51). Conversely it is plausible that the  $\ell \gg r$  limit can be derived by neglecting the first term on the left hand side of eq (49). In that case we are simply solving Poisson's equation in ordinary three dimensional space for the potential of a point charge  $\lambda/\ell$  thereby justifying eq (50).

*Kaluza-Klein modes and a resonance.* It is also interesting to consider the solutions to the DGP model in the absence of sources. Remarkably it is found there is no zero mode but there is a continuum of massive Kaluza-Klein modes and the mass parameter  $\mu$  can take any positive real value. To demonstrate this result we seek solutions to eq (49) of the form  $v(w) \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(-i\omega t)$ . Substituting this ansatz into eq (49) we obtain

$$-\frac{1}{2} \frac{d^2}{dw^2} v - \frac{\ell \mu^2}{2} \delta(w) v = \frac{\mu^2}{2} v. \quad (52)$$

Here as usual we have written  $\omega^2 = k^2 + \mu^2$ . Superficially this resembles the Schrödinger equation for a non-relativistic particle moving in one dimension under the influence of an attractive delta function potential. The analogy is not perfect because the strength of the delta

potential depends on  $\frac{1}{2}\mu^2$  which is analogous to the “energy”. Nonetheless we can draw upon experience with delta potentials in quantum mechanics in solving eq (52). We expect a solution of the scattering form

$$\begin{aligned} v_+(w; \mu) &= \frac{1}{\sqrt{2\pi}} \exp(i\mu w) + \frac{r(\mu)}{\sqrt{2\pi}} \exp(-i\mu w) \text{ for } w < 0 \\ &= \frac{t(\mu)}{\sqrt{2\pi}} \exp(i\mu w) \text{ for } w > 0. \end{aligned} \quad (53)$$

There should be a second solution,  $v_-(w; \mu)$ , corresponding to a plane wave incident along the negative  $w$  axis. The detailed analysis of these solutions is relegated to the problems. Here we note that the “transmission” coefficient is found to be

$$|t(\mu)|^2 = \frac{1}{1 + (\mu\ell/2)^2}. \quad (54)$$

There is a peak in the transmission at  $\mu = 0$ . The peak has a Lorentzian form and a width of  $1/\ell$ . Thus, although there is no discrete zero-mode bound to the brane, there is a resonance in the Kaluza-Klein modes at  $\mu = 0$  [31].

*Heart of Darkness.* A number of lines of evidence, starting with supernova red-shift surveys [22], have led to the discovery that the geometry of the Universe is flat FRW (see Appendix A) and that the expansion of the Universe is accelerating [8]. The effect can be parameterized by positing that the Universe is uniformly filled with a mysterious “dark energy”. Little is known about dark energy except that its pressure is negative and equal in magnitude to its energy density (by contrast a photon gas has a positive pressure equal to one-third of its energy density). Einstein's equations show that if the Universe was filled uniformly with dark energy at the appropriate density it would undergo exponential expansion. This density is denoted  $\rho_{\text{crit}}$  and it is given by  $\rho_{\text{crit}} = 3c^2 H^2 / 8\pi G$ . Here  $H$  is the Hubble parameter, a measure of the present expansion rate of the Universe. A host of observations, including supernova red-shifts, cosmic microwave background anisotropies, and galaxy cluster surveys, are consistent with the total energy density of the Universe being essentially equal to  $\rho_{\text{crit}}$  and with dark energy constituting a fraction  $\Omega_\Lambda \approx 0.7$  of the total energy density. The DGP model was introduced originally as an alternative explanation for the accelerated expansion of the Universe. In the DGP model there is no dark energy. Instead gravity becomes modified on long distance scales. As the above discussion suggests, on distance scales short compared to  $\ell$  space-time appears four-dimensional: gravity obeys Newton's inverse square law and is strong. On distance scales long compared to the screening length  $\ell$  the five dimensionality of space-time is revealed: the gravitational force now falls off much more rapidly, as the inverse cube of the distance, making gravity less effective on cosmological scales. Using dimensional analysis, the required screening length  $\ell = c^2 / \sqrt{\rho_\Lambda G}$ . Making use of the previous relations between  $\rho_\Lambda$  and  $\rho_{\text{crit}}$  we find the screening length

$\ell \sim c/H\sqrt{\Omega_\Lambda} \sim 10^9$  light-years, comparable to the size of the observable Universe.

## VI. CONCLUSION

The standard model of particle physics is believed to be a low energy approximation to a more fundamental theory that remains to be discovered. According to the grand unification paradigm [23] the strong and electroweak forces are unified at the stupendous energy scale  $E_{\text{GUT}} \sim 10^{25}$  eV  $\sim 10^{-3}E_P$ . The standard model, or a supersymmetric extension of it, is presumed valid up to this scale. Supersymmetry postulates the existence of a large number of new particles, super-partners of the known particles, with masses not much larger than those of the known particles. Colliders may be able to discover the predicted new particles but grand unification remains beyond the reach of any collider of the near future. Physics on the grand unified scale is accessible only to precision measurements such as proton decay or via cosmology. Crucially within this paradigm the grand unified scale is well separated from the Planck scale and gravity is a separate problem not addressed by grand unification. Extra dimensional models represent a very different view of the world. As discussed above, according to the ADD model, the fundamental scale for the unification of all forces, including gravity, is the weak scale, not  $E_{\text{GUT}}$  or  $E_P$ . According to the Randall Sundrum model too, although the fundamental scale for gravity is the Planck scale, quantum gravity effects are accessible to colliders in the form of TeV scale Kaluza-Klein particles. In section IV we have discussed some ongoing experiments that will determine which of these scenarios, if any, corresponds to reality. The ADD and Randall-Sundrum models are concerned with the modification of gravity on short distance scales (the ultraviolet limit). By contrast the DGP model is concerned with gravity on cosmological scales (the infrared limit). The DGP model is now disfavored by observations of the accelerated expansion of the Universe [24] but it has many descendants that are thriving [9]. On the theoretical front a recent development in string theory is the AdS-CFT correspondence that demonstrates a deep relationship between theories defined in the bulk of AdS space-time and on its boundary [2]. This correspondence further illuminates the Randall Sundrum model. Many interesting applications of the AdS-CFT correspondence are in condensed matter physics [25]. Thus extra dimensional models sit at a vibrant intersection of theory and experiment and different subfields of physics.

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## Appendix A: Differential Geometry

*Curved space.* Let us start with ordinary three dimensional Euclidean space. Adopting cartesian co-ordinates

the distance between neighboring points is given by the Pythagorean formula  $ds^2 = dx^2 + dy^2 + dz^2$ . Suppose however we use spherical polar co-ordinates  $(r, \theta, \varphi)$  to label points in the same space. The distance between neighboring points in spherical polar co-ordinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \quad (\text{A1})$$

Eq (A1) follows readily from the transformation that given Cartesian co-ordinates in terms of polar; namely,  $z = r \cos \theta$ ,  $x = r \sin \theta \cos \varphi$  and  $y = r \sin \theta \sin \varphi$ . Regardless of the co-ordinates we use, the geometry of the space is fully specified by the formula for distance between neighboring points.

Thus far we are only describing the same flat space in different co-ordinates. Now let us discuss curved spaces. The simplest example is the surface of a sphere of radius  $R$  that is located at the origin of the Cartesian co-ordinate system. Points on the surface of the sphere can be labelled by the co-latitude  $\theta$  and the longitude  $\varphi$ . The distance between neighboring points on the surface of a sphere is given by

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2. \quad (\text{A2})$$

The co-ordinates are restricted to lie in the range  $0 \leq \theta \leq \pi$  and  $0 \leq \varphi < 2\pi$ . Although it is helpful to picture the sphere as a surface in three dimensional Euclidean space mathematically the space is fully defined by eq (A2) together with a statement about the allowed range of the co-ordinates.

Another example of a curved space is the two dimensional hyperbolic space. Points in this space are labelled by the co-ordinates  $(\theta, \varphi)$  which have the ranges  $0 \leq \theta < \infty$  and  $0 \leq \varphi < 2\pi$ . The distance between neighboring points on the surface of hyperbolic space is given by

$$ds^2 = R^2 d\theta^2 + R^2 \sinh^2 \theta d\varphi^2. \quad (\text{A3})$$

Hyperbolic space cannot be regarded as a surface in ordinary three dimensional space but it can be embedded in a different flat three dimensional space in which the co-ordinates  $(x, y, z)$  have the usual range  $-\infty < x, y, z < \infty$  but the distance between neighboring points is given by

$$ds^2 = -dz^2 + dx^2 + dy^2 \quad (\text{A4})$$

This space is flat but it has the peculiar feature that the square of the distance between nearby points can be positive, negative or zero. Mathematically this space is said to have an indefinite metric. The hyperbolic space is the two dimensional surface that satisfies the equation

$$z^2 - x^2 - y^2 = R^2 \quad (\text{A5})$$

together with the condition  $z > 0$ . It is easy to see that the constraint eq (A5) is automatically satisfied if we write

$$z = R \cosh \theta, x = R \sinh \theta \cos \varphi, y = R \sinh \theta \sin \varphi. \quad (\text{A6})$$

Furthermore the entire hyperboloid will be covered by this parametrization if  $(\theta, \varphi)$  are allowed the range noted above. It is now a simple exercise to see that the distance formula eq (A3) follows when eq (A6) is substituted into eq (A4).

*The Laplacian.* Let us now return to ordinary three dimensional space inhabited by a scalar field such as the scalar potential,  $\phi(x, y, z)$ . It is a familiar result of elementary calculus that the gradient of a scalar field is a vector; and that the divergence of the gradient, which is the Laplacian, is a scalar. In symbols,  $\nabla \cdot \nabla \phi = \nabla^2 \phi$  is a scalar. In Cartesian co-ordinates the Laplacian may be written as

$$\nabla^2 \phi = \frac{\partial^2}{\partial x^2} \phi + \frac{\partial^2}{\partial y^2} \phi + \frac{\partial^2}{\partial z^2} \phi. \quad (\text{A7})$$

It is helpful sometimes to work in a different co-ordinate system such as polar or cylindrical co-ordinates and therefore it is useful to be able to express the Laplacian directly in other co-ordinate systems. Suppose we transform to a system of co-ordinates  $(\xi, \eta, \zeta)$  in which the distance between neighboring points is given by

$$ds^2 = h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 + h_\zeta^2 d\zeta^2. \quad (\text{A8})$$

The distance formula eq (A8) can easily be derived from the Pythagorean formula  $ds^2 = dx^2 + dy^2 + dz^2$  given  $(\xi, \eta, \zeta)$  as functions of  $(x, y, z)$ . In general the formula might involved cross terms like  $d\xi d\eta$ ,  $d\eta d\zeta$  and  $d\zeta d\xi$  but here we will focus only on orthogonal co-ordinates in which such terms are absent. Most co-ordinate systems of interest in mathematical physics are orthogonal. The factors  $h_\xi, h_\eta$  and  $h_\zeta$  are called scale factors and their product  $h = h_\xi h_\eta h_\zeta$  is called the invariant measure. For example for spherical polar co-ordinates the scale factors are  $h_r = 1, h_\theta = r, h_\varphi = r \sin \theta$  and the invariant measure  $h = r^2 \sin \theta$ . Now it is proved in books on electromagnetism and mathematical physics that the Laplacian in the co-ordinates  $(\xi, \eta, \zeta)$  is given by

$$\frac{1}{h} \frac{\partial}{\partial \xi} \left( \frac{h}{h_\xi^2} \frac{\partial}{\partial \xi} \phi \right) + \frac{1}{h} \frac{\partial}{\partial \eta} \left( \frac{h}{h_\eta^2} \frac{\partial}{\partial \eta} \phi \right) + \frac{1}{h} \frac{\partial}{\partial \zeta} \left( \frac{h}{h_\zeta^2} \frac{\partial}{\partial \zeta} \phi \right) \quad (\text{A9})$$

The generalization of this result to more or fewer dimensions should be obvious. It is a good exercise for the reader to use eq (A9) construct the Laplacian in familiar cases like spherical polar and cylindrical co-ordinates.

Thus far we have been discussing the mundane subject of writing the Laplacian in different co-ordinate systems in ordinary flat three dimensional space. Remarkably it turns out that in a curved space in which the distance formula has the form eq (A8) it is still true that the Laplacian is given by eq (A9). This is intuitively plausible and gives us a simple method to construct the Laplacian in curved space without going through the full machinery of differential geometry [32]. It is a good exercise for the reader to use eq (A9) to write an expression for the Laplacian on the surface of a sphere and in hyperbolic space.

*Curved space-time.* The central dogma of special relativity is that an inertial observer using Cartesian co-ordinates can label events by the co-ordinates  $(t, x, y, z)$  which lie in the range  $-\infty < t, x, y, z < \infty$  and that the interval between two nearby events is given by eq (1). Suppose we transform to a system of co-ordinates  $(\tau, \xi, \eta, \zeta)$  in which the space-time interval between neighboring events is given by

$$ds^2 = h_\tau^2 d\tau^2 - h_\xi^2 d\xi^2 - h_\eta^2 d\eta^2 - h_\zeta^2 d\zeta^2. \quad (\text{A10})$$

The interval eq (A10) can easily be derived from eq (1) given  $(\tau, \xi, \eta, \zeta)$  as a function of  $(t, x, y, z)$ . In general the formula might involve cross terms like  $d\tau d\xi$  or  $d\xi d\eta$  but here we will focus only on orthogonal co-ordinates in which such terms are absent. The factors  $h_\tau, h_\xi, h_\eta$  and  $h_\zeta$  are called scale factors and their product  $h = h_\tau h_\xi h_\eta h_\zeta$  is called the invariant measure. Thus far we are dealing with the Minkowski space-time of special relativity. Although eq (A10) looks complicated we know that the underlying space-time is flat and that we can always transform back to Cartesian co-ordinate  $(t, x, y, z)$  in which the interval has the simple form eq (1). By contrast a curved space-time is one in which the space-time interval might have a form like eq (A10) but where it is impossible to find an alternative set of co-ordinates in which the interval globally has the simple flat form given in eq (1).

A concrete example of a curved space-time is the Friedman-Walker-Robertson space-time that is believed to describe our expanding universe. It has the space-time interval

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \quad (\text{A11})$$

Here  $a(t)$  is called the scale factor and it grows with time reflecting the expansion of the Universe. This space-time is called flat FRW. Notwithstanding that name it is undeniably a curved space-time. There is no change of co-ordinates which will bring the space-time interval (A11) to the form eq (1).

A second example of a curved space-time is the celebrated Anti-de-Sitter (AdS) space-time. Like the hyperbolic space discussed above, AdS space-time is best understood by embedding it in a space of higher dimensionality. We start with a six dimensional flat space with the interval

$$ds^2 = du^2 + dv^2 - d\alpha^2 - d\beta^2 - d\eta^2 - d\xi^2. \quad (\text{A12})$$

The co-ordinates have the usual range  $-\infty < u, v, \alpha, \beta, \eta, \xi < \infty$ . AdS is the five dimensional surface defined by the constraint

$$u^2 + v^2 - \alpha^2 - \beta^2 - \eta^2 - \xi^2 = \frac{1}{\gamma^2}. \quad (\text{A13})$$

We now put down new co-ordinates  $(\zeta, t, x, y, z)$  on the half of AdS space-time that satisfies  $v + \xi > 0$ . The new co-ordinates are related to the old via

$$v + \xi = \frac{1}{\gamma\zeta}, \quad u = \frac{1}{\gamma\zeta}, \quad \alpha = \frac{x}{\gamma\zeta}, \quad \beta = \frac{y}{\gamma\zeta}, \quad \eta = \frac{z}{\gamma\zeta}. \quad (\text{A14})$$

The constraint eq (A13) is automatically satisfied provided we take

$$v - \xi = \frac{1}{\gamma\zeta} (\zeta^2 + x^2 + y^2 + z^2 - t^2). \quad (\text{A15})$$

The new co-ordinates have the range  $\zeta \geq 0$  and  $-\infty < t, x, y, z, < \infty$ . Making use of eq (A14) and eq (A12) it follows after some algebra that the interval on AdS space-time in terms of the new co-ordinate is given by

$$ds^2 = \frac{1}{\gamma^2\zeta^2} (dt^2 - d\zeta^2 - dx^2 - dy^2 - dz^2), \quad (\text{A16})$$

exactly the form for Randall-Sundrum space-time in conformal co-ordinates.

*Scalar waves in curved space-time.* A scalar wave in Minkowski space-time obeys the wave eq (2). The operator  $\square^2$  defined in eq (2) is called the d'Alembertian and is the space-time analog of a Laplacian. If  $\phi$  is a scalar then so is  $\square^2\phi$  (see for example, ref [14], vol II, chapter 25, section 3). Suppose we now transform to a system of co-ordinates  $(\tau, \xi, \eta, \zeta)$  in which the space-time interval between neighboring events is given by eq (A10). Evidently in this co-ordinate system the d'Alembertian operator will have the form

$$\begin{aligned} \square^2\phi = & \frac{1}{h} \frac{\partial}{\partial\tau} \left( \frac{h}{h_\tau^2} \frac{\partial\phi}{\partial\tau} \right) - \frac{1}{h} \frac{\partial}{\partial\xi} \left( \frac{h}{h_\xi^2} \frac{\partial\phi}{\partial\xi} \right) \\ & - \frac{1}{h} \frac{\partial}{\partial\eta} \left( \frac{h}{h_\eta^2} \frac{\partial\phi}{\partial\eta} \right) - \frac{1}{h} \frac{\partial}{\partial\zeta} \left( \frac{h}{h_\zeta^2} \frac{\partial\phi}{\partial\zeta} \right) \end{aligned} \quad (\text{A17})$$

Thus far we are discussing the mundane subject of writing the d'Alembertian in different co-ordinate systems in ordinary flat Minkowski space-time. Remarkably it turns out that in a curved space-time in which the space-time interval has the form eq (A10), it is still true that the d'Alembertian is given by eq (A17). This is intuitively plausible and it gives us a simple method to write down the wave equation in curved space-times without going through the full machinery of differential geometry. Although we have written our results specifically for a four dimensional space-time the generalization to higher or lower dimensions is self-evident.

*Big bang for your buck.* For practice, and as a return on the time invested in mastering this appendix, consider a scalar field that obeys an equation that is a slight generalization of the ordinary wave equation. Since  $\square^2\phi$  is a scalar we can add to it any function of  $\phi$  (because  $\phi$  and any function of it are also scalars). Thus we consider the wave equation  $\square^2\phi + V'(\phi) = 0$ . Here  $V$  is a specified function of  $\phi$  called the potential and the prime denotes differentiation with respect to  $\phi$ .  $V = 0$  corresponds to the ordinary wave equation;  $V = \frac{1}{2}\mu^2\phi^2$  corresponds to the Klein-Gordon equation. The reader should verify by using eq (A17) that for flat FRW space-time the wave equation would have the form

$$\frac{\partial^2\phi}{\partial t^2} + \frac{3}{a} \frac{\partial a}{\partial t} \frac{\partial\phi}{\partial t} - \frac{1}{a^2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{dV}{d\phi} = 0. \quad (\text{A18})$$

This is a central equation of inflationary cosmology. It describes the evolution of the ‘‘inflation scalar field’’  $\phi$  in a flat FRW space-time.  $V$  is called the inflation potential.

## Appendix B: One dimensional quantum mechanics

*Normalization.* We begin by recalling some useful results from undergraduate quantum mechanics. For derivations readers should consult their favorite textbook. Consider a free particle in one dimension. In a state of definite momentum the state of the particle is described by the wave function

$$\psi(x; k) = \frac{1}{\sqrt{2\pi}} \exp(ikx). \quad (\text{B1})$$

The pre-factor has been chosen to ensure the normalization

$$\int_{-\infty}^{\infty} dx \psi^*(x; p) \psi(x; k) = \delta(p - k). \quad (\text{B2})$$

Now suppose that there is a potential barrier  $V(x)$  that is localized near the origin. Sufficiently far from the origin the particle is free and we expect to find scattering solutions of the form

$$\begin{aligned} \psi(x; k) &= \frac{1}{\sqrt{2\pi}} \exp(ikx) + \frac{r}{\sqrt{2\pi}} \exp(-ikx) \\ &\quad \text{for } x \rightarrow -\infty \\ &= \frac{t}{\sqrt{2\pi}} \exp(ikx) \quad \text{for } x \rightarrow \infty \end{aligned} \quad (\text{B3})$$

Here the reflection coefficient  $r$  and the transmission coefficient  $t$  might depend on the wave-vector  $k$  and satisfy the unitarity condition  $|r|^2 + |t|^2 = 1$ . The scattering wave functions still satisfy the normalization condition eq (B2). The solution above corresponds to the scattering of a particle that is incident on the barrier from the left. An analogous solution may be written down for particles incident from the right but will not be needed here. Now suppose that the potential  $V \rightarrow \infty$  as  $x \rightarrow \infty$  or else that the particle encounters an impenetrable barrier such as a hard wall at the origin. In that case the transmission coefficient  $t \rightarrow 0$  and the reflection coefficient has magnitude unity and may be written as  $r = \exp(i2\delta)$ . This defines the scattering phase shift  $\delta$  which may depend on  $k$ . For this circumstance the scattering solution has the form

$$\psi(x; k) = \sqrt{\frac{2}{\pi}} \cos(kx - \delta) \quad (\text{B4})$$

This form is obtained by substituting  $r \rightarrow \exp(i2\delta)$  in eq (B3) and multiplying the solution by a factor of  $\exp(-i\delta)$ . The scattering solutions eq (B4) still satisfy the normalization condition eq (B2).

*Bessel solution.* We turn now to the solution to eq (23) for  $\mu > 0$ . It is convenient to transform to the dependent

variable  $\varphi$  defined by  $\psi = \zeta^{1/2}\varphi$  and to change to the independent variable  $\xi = \mu\zeta$ . In terms of these variables eq (23) is revealed to be Bessel's equation of the second order

$$\frac{d^2\varphi}{d\xi^2} + \frac{1}{\xi}\frac{d\varphi}{d\xi} + \left(1 - \frac{4}{\xi^2}\right)\varphi = 0 \quad (\text{B5})$$

with independent solutions  $J_2(\xi)$  and  $Y_2(\xi)$ . Hence the general solution to eq (23) has the form given in eq (25) with the coefficients  $\alpha$  and  $\beta$  at this stage arbitrary. Imposing the boundary condition  $\psi = -\frac{2}{3}\gamma^{-1}\psi'$  on  $\psi$  at  $\zeta = \gamma^{-1}$  determines the ratio  $\alpha/\beta$ ,

$$\frac{\alpha}{\beta} = -Y_1\left(\frac{\mu}{\gamma}\right)/J_1\left(\frac{\mu}{\gamma}\right), \quad (\text{B6})$$

and leads to eq (26). Here we have used the Bessel function recursion  $xZ_1(x) = 2Z_2(x) + xZ_2'(x)$  where  $Z$  denotes either  $J$  or  $Y$ . Finally let us write

$$\alpha = \sqrt{\alpha^2 + \beta^2} \cos \Delta \quad \text{and} \quad \beta = \sqrt{\alpha^2 + \beta^2} \sin \Delta \quad (\text{B7})$$

which defines  $\Delta$ . Inserting this form into the solution eq (25) and making use of the large argument asymptotics of the Bessel and Neumann functions,

$$J_2(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{5}{4}\pi\right), \quad Y_2(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{5}{4}\pi\right), \quad (\text{B8})$$

we obtain

$$\psi(\zeta) \approx \sqrt{\frac{2(\alpha^2 + \beta^2)}{\pi\mu}} \cos\left(\mu\zeta - \frac{5}{4}\pi - \Delta\right) \quad (\text{B9})$$

for  $\mu\zeta \gg 1$ . Comparing eq (C9) to eq (B4) we see that our solutions will have the desired normalization, eq (27), if we choose  $\alpha^2 + \beta^2 = \mu$ . This completes the derivation of the continuum solutions to eq (23).

### Appendix C: Problems.

*Problem 1.* What data would you invoke in order to rule out the ADD model with a single extra dimension? (*Answer:*  $10^{13}$  m is comparable to the size of Pluto's orbit. An extra dimension of this size would lead to violations of Newton's inverse square law of gravity throughout the solar system. Tycho Brahe's data, and very likely even Sumerian tablets, would be sufficient to rule out such violations.)

*Problem 2. The ADD hierarchy problem.* (a) An alternative way to describe the hierarchy problem is in terms of length scales. (i) Construct a length scale out of the three fundamental constants  $c, G$  and  $\hbar$ . This is the Planck length,  $\ell_P$ . (ii) Denote the electroweak scale  $\eta = 1$  TeV. From  $\eta, \hbar$  and  $c$  construct the length scale  $\ell_{ew}$  that corresponds to electroweak physics. The conventional hierarchy problem is then the observation that

$\ell_P \ll \ell_{ew}$ . (b) The ADD model solves this hierarchy problem by introducing extra dimensions of characteristic scale  $L$ . Compare  $L$  to  $\ell_{ew}$  and comment.

*Answer 2.* (a)  $\ell_P = \sqrt{\hbar G/c^3} \sim 10^{-35}$  m.  $\ell_{ew} = \hbar c/\eta \sim 10^{-19}$  m. (b) In section III we found that  $L \sim 1$  mm if there are two extra dimensions. Thus the ADD model resolves the hierarchy  $\ell_P \ll \ell_{ew}$  by introducing extra dimensions of length scale  $L$ ; however it suffers from a hierarchy problem of its own, namely,  $\ell_{ew} \ll L$ .

*Problem 3. Three dimensional sphere.* Points in flat four dimensional space can be labelled by the four Cartesian co-ordinates  $(x, y, z, w)$  that have the range  $-\infty < x, y, z, w < \infty$ . The distance between neighboring points is given by  $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$ . The three dimensional sphere of radius  $R$  is the set of points that satisfy the constraint  $x^2 + y^2 + z^2 + w^2 = R^2$ . Points on the three dimensional sphere can be specified by the coordinates  $(\psi, \theta, \varphi)$  that are related to Cartesian co-ordinates via

$$\begin{aligned} w &= R \cos \psi; \\ z &= R \sin \psi \cos \theta; \\ x &= R \sin \psi \sin \theta \cos \varphi; \\ y &= R \sin \psi \sin \theta \sin \varphi. \end{aligned} \quad (\text{C1})$$

Here the co-latitudes have the range  $0 \leq \psi, \theta \leq \pi$  while the longitude has the range  $0 \leq \varphi < 2\pi$ . (a) Justify the parametrization (C1) and explain the ranges on the coordinates  $(\psi, \theta, \varphi)$ . (b) Determine the distance between two nearby points with co-ordinates  $(\psi, \theta, \varphi)$  and  $(\psi + d\psi, \theta + d\theta, \varphi + d\varphi)$  in terms of  $(\psi, \theta, \varphi)$  and  $d\psi, d\theta$  and  $d\varphi$ .

*Answer 3.* (a) The condition  $x^2 + y^2 + z^2 + w^2 = R^2$  implies that  $-R \leq w \leq R$ . Hence we may write  $w = R \cos \psi$  with  $0 \leq \psi \leq \pi$ . The constraint now reads  $x^2 + y^2 + z^2 = R^2 \sin^2 \psi$  which implies that  $-R \sin \psi \leq z \leq R \sin \psi$ . Hence we may write  $z = R \sin \psi \cos \theta$  with  $0 \leq \theta \leq \pi$ . The constraint now reads  $x^2 + y^2 = R^2 \sin^2 \psi \sin^2 \theta$ , the equation of a circle of radius  $R \sin \psi \sin \theta$ . Obviously we can now write  $x = R \sin \psi \sin \theta \cos \varphi$  and  $y = R \sin \psi \sin \theta \sin \varphi$  with  $\varphi$  the angle around the circle and hence lying in the range  $0 \leq \varphi < 2\pi$ .

(b)  $ds^2 = R^2 d\psi^2 + R^2 \sin^2 \psi d\theta^2 + R^2 \sin^2 \psi \sin^2 \theta d\varphi^2$ .

*Problem 4. Three dimensional hyperbolic space.* Points in a flat four dimensional space can be labelled by the four Cartesian co-ordinates  $(x, y, z, w)$  that have the range  $-\infty < x, y, z, w < \infty$ . The space has an indefinite metric. The distance between neighboring points is given by  $ds^2 = dx^2 + dy^2 + dz^2 - dw^2$ . The three dimensional hyperbolic space is defined as the set of points that satisfy the constraints  $w^2 - z^2 - x^2 - y^2 = R^2$  and  $w > 0$ . Points on three dimensional hyperbolic space can be labelled by the co-ordinates  $(\psi, \theta, \varphi)$  that are related to the Cartesian co-ordinates via

$$\begin{aligned} w &= R \cosh \psi; \\ z &= R \sinh \psi \cos \theta; \\ x &= R \sinh \psi \sin \theta \cos \varphi; \\ y &= R \sinh \psi \sin \theta \sin \varphi. \end{aligned} \quad (\text{C2})$$

Here the co-ordinates have the ranges  $0 \leq \psi < \infty$ ,  $0 \leq \theta \leq \pi$  and  $0 \leq \varphi < 2\pi$ . (a) Justify the parametrization (C2) and explain the ranges on the coordinates  $(\psi, \theta, \varphi)$ . (b) Determine the distance between two nearby points with co-ordinates  $(\psi, \theta, \varphi)$  and  $(\psi + d\psi, \theta + d\theta, \varphi + d\varphi)$ .

*Answer 4.* (a) The constraints that  $w^2 - x^2 - y^2 - z^2 = R^2$  and  $w > 0$  together imply that  $w > R$ . Hence we may write  $w = R \cosh \psi$  with  $0 \leq \psi < \infty$ . Adopting this form the constraint then becomes  $x^2 + y^2 + z^2 = R^2 \sinh^2 \psi$ . From here on the reasoning is the same as problem 2(a). (b)  $ds^2 = R^2 d\psi^2 + R^2 \sinh^2 \psi d\theta^2 + R^2 \sinh^2 \psi \sin^2 \theta d\varphi^2$ .

*Problem 5. DGP Analysis.* Here we fill in the steps that lead from eq (49) to eqs (50) and (51). (a) Let  $\tilde{g}(p)$  be the Fourier transform of  $g(w)$ . Show that the Fourier transform of  $g(w)\delta(w)$  is a constant given by

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \tilde{g}(p). \quad (\text{C3})$$

Thus it follows that the Fourier transform of  $-\delta(w)\nabla^2 \phi(x, y, z, w)$  is  $k^2 \tilde{f}(\mathbf{k})$  where

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \tilde{\phi}(\mathbf{k}, p) \quad (\text{C4})$$

and  $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ . (b) Use eq (C4) to rewrite eq (49) in Fourier space. You should obtain

$$\tilde{\phi}(\mathbf{k}, p) + \frac{\ell k^2}{k^2 + p^2} \tilde{f}(\mathbf{k}) = \frac{\lambda}{k^2 + p^2}. \quad (\text{C5})$$

(c) Use eqs (C4) and (C5) to determine  $\tilde{f}$ . You should obtain

$$\tilde{f}(\mathbf{k}) = \frac{\lambda}{2k} \frac{1}{(1 + \frac{1}{2}\ell k)}. \quad (\text{C6})$$

(d) Show that it is sufficient to know  $\tilde{f}$  to determine  $\phi(\mathbf{r}, 0)$ . One does not need  $\tilde{\phi}$ . You should find

$$\phi(\mathbf{r}, 0) = \int \frac{d\mathbf{k}}{(2\pi)^3} \tilde{f}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}). \quad (\text{C7})$$

(e) The angular integrals in eq (C7) can be exactly evaluated. Perform the angular integration. You should obtain

$$\phi(\mathbf{r}, 0) = \frac{\lambda}{4\pi^2 r} \int_0^\infty dk \frac{\sin(kr)}{(1 + \frac{1}{2}\ell k)}. \quad (\text{C8})$$

This is the exact expression for the potential of a point source in the DGP model. It can be rewritten in terms of rather obscure special functions (the cosine-integral and sine-integral functions) but those expressions are not especially edifying. (f) Verify that eq (C8) matches eqs

(51) and (50) in the appropriate limits. Useful asymptotic formulae:

$$\mathcal{I}(\alpha) = \int_0^\infty dx \frac{\sin x}{1 + \alpha x} \quad (\text{C9})$$

has the asymptotic behavior  $\mathcal{I}(\alpha) \approx 1 - 2!\alpha^2 + 4!\alpha^4 - 6!\alpha^6 + \dots$  for small  $\alpha$  and  $\mathcal{I}(\alpha) \approx \pi/2\alpha + [\gamma_E - 1 + \ln(1/\alpha)](1/\alpha^2) + \dots$  for large  $\alpha$ . Here  $\gamma_E = 0.577216\dots$  is Euler's constant.

*Problem 6. RS1 Kaluza-Klein mode analysis.* (a) Verify that the zero-mode solution  $\phi = g(\zeta, \mathbf{r}; \mathbf{k}) \exp(i\omega t)$  satisfies the wave equation (21) as well as the boundary condition  $\partial\phi/\partial\zeta = 0$  at both  $\zeta_l = \gamma^{-1}$  (left brane) and  $\zeta_r = \gamma^{-1} \exp(\gamma\ell)$  (right brane). Here  $g$  is given by eq (29) and  $\psi_0 = \mathcal{A}\zeta^{-3/2}$  where  $\mathcal{A}$  is a suitable normalization constant. (b) The Kaluza-Klein mode  $\phi = f(\zeta, \mathbf{r}; \mathbf{k}) \exp(i\omega t)$ , with  $f$  given by eq (30) already satisfies the wave equation (21) as well as the boundary condition on the left brane. Impose the boundary condition  $\partial\phi/\partial\zeta = 0$  on the right brane,  $\zeta = \gamma^{-1} \exp(\gamma\ell)$  to obtain the quantization condition that must be satisfied by the allowed values of  $\mu$ . You should find

$$Y_1\left(\frac{\mu}{\gamma}\right) J_1\left(\frac{\mu}{\gamma} e^{\gamma\ell}\right) = J_1\left(\frac{\mu}{\gamma}\right) Y_1\left(\frac{\mu}{\gamma} e^{\gamma\ell}\right). \quad (\text{C10})$$

(c) Simplify eq (C10) assuming that  $(\mu/\gamma)e^{\gamma\ell} \gg 1$  and  $\mu/\gamma \ll 1$ . Use the resulting expression to determine approximate quantized values of  $\mu$ . You should obtain

$$\cot\left(\frac{\mu}{\gamma} e^{\gamma\ell} - \frac{3}{4}\pi\right) \approx 0. \quad (\text{C11})$$

for the approximate quantization condition and  $\mu_n = \gamma \exp(-\gamma\ell)(n\pi + \frac{\pi}{4})$  with  $n = 0, 1, 2, \dots$  for the quantized  $\mu$  values. Rigorously the approximations made in this part are valid only for large  $n$  but in fact these  $\mu$  values are surprisingly accurate for all values of  $n$ .

*Problem 7. Klein-Gordon field on RS1 right brane.* (a) Let us work with the original co-ordinates  $(t, x, y, z, w)$  on Randall-Sundrum space-time. In these co-ordinates the space-time interval is given by eq (19). As a warm up write the Klein-Gordon equation  $\square^2 \phi + \mu^2 \phi = 0$  for a scalar field  $\phi$  that lives in the bulk. (b) Now let us consider scalar field that is confined to the right brane. The space-time interval between neighboring points on the right brane is given by

$$ds^2 = e^{-2\gamma\ell}(dt^2 - dx^2 - dy^2 - dz^2). \quad (\text{C12})$$

Thus the scale factors  $h_t = h_x = h_y = h_z = e^{-\gamma\ell}$  and the invariant measure  $h = e^{-4\gamma\ell}$ . Use eq (C12) to write the Klein-Gordon equation  $\square_R^2 \xi + m^2 \xi = 0$  for a field  $\xi$  confined to the right brane. You should obtain eq (43).

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  - [26] Evaluate the Fourier transform of  $f(x)$  defined as  $f(x) = \exp(-\mu x)$  for  $x > 0$  and  $f(x) = -\exp(-\mu x)$  for  $x < 0$ . This is readily found to be  $-2ik/(k^2 + \mu^2)$ . Now write  $f$  in terms of its Fourier transform, take the real part of both sides, and note that the Fourier integral over  $k$  is even.
  - [27] Randall and Sundrum did not merely assume this form but rather showed that it was compatible with general relativity if they made particular assumptions about the brane. However that analysis is beyond the scope of this article.
  - [28] To understand how a boundary condition is essentially a contact interaction consider a particle moving in one dimension free except for a delta function potential at the origin  $\lambda^{-1}\delta(x)$ . If we look for solutions that are symmetric about the origin we need to impose the boundary condition  $\psi'(0) = \lambda^{-1}\psi(0)$ . (This can be derived by integrating the Schrödinger equation across the origin as discussed in textbook treatments of the delta function model).
  - [29] Recall that  $v = \zeta^{3/2}\psi$  and  $\phi = v(\zeta) \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(i\omega t)$ . Transforming the boundary condition on  $\psi$  into a boundary condition on  $v$  we see that  $\phi$  simply satisfies Neumann boundary conditions on the brane,  $\partial\phi/\partial\zeta|_{\zeta \rightarrow \gamma^{-1}} = 0$ .
  - [30] The identity is merely a precise statement that the modes  $g$  and  $f$  are a complete basis. It is easily proved by making use of the explicit forms of  $g$  and  $f$  in eqs (29), (30), as well as the completeness relation (28) and the standard Fourier integral  $\int d\mathbf{k} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')] = (2\pi)^3 \delta(\mathbf{r} - \mathbf{r}')$ .
  - [31] As discussed in introductory textbooks, a resonance is an almost-bound state in quantum mechanics. As a concrete example consider a non-relativistic particle moving in one dimension in a double barrier potential. If the barrier walls were infinitely high there would be bound states between the barriers. For finite barriers these states are no longer bound and become resonances. They acquire a finite lifetime as the particle can escape by tunneling. If we consider scattering states in which a particle is incident on the double barrier potential from outside we find sharp Lorentzian peaks in the transmission coefficient as a function of the energy. The energy of the peaks corresponds roughly to the energy of the resonance and the width of the peak to the lifetime.
  - [32] A full course in differential geometry derives the expression for a Laplacian by first generalizing the notions of scalars, vectors and tensors to curved spaces, developing the notion of a covariant derivative, and then defining the Laplacian as the scalar obtained by contracting a second covariant derivative of a scalar field.